

$$\Rightarrow \frac{df}{d\eta} \rightarrow C[f] \rightarrow \theta(\eta) \rightarrow S \rightarrow \frac{\partial}{\partial \eta} (\theta e^{ik_\mu \eta - z}) \rightarrow \bar{S} \rightarrow j_s \rightarrow \theta_s(\eta)$$

$$P(\vec{n}, \eta) = \frac{1}{e^{P/T(\eta) [1 + \theta(\vec{n}, \eta)]}}$$

$$\frac{df}{d\eta} = a \cdot C[f] \quad C = h = k_b = 1$$

$$\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \eta^i \frac{\partial f}{\partial x^i} - P \frac{\partial f}{\partial p} \left(H + \frac{\partial \Phi}{\partial \eta} + \eta^i \frac{\partial \Psi}{\partial x^i} \right)$$

$$\text{zero-order} \quad \Phi = \Psi = 0 \quad \frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \eta^i \frac{\partial f}{\partial x^i} - P \frac{\partial f}{\partial p} H$$

$$\text{first-order} \quad f = f_0 - P \frac{\partial f_0}{\partial p} \theta \quad e^{-1}$$

$$P(\vec{n}, \eta) \equiv \frac{\sigma_T}{T} \quad \frac{df}{d\eta} = C[f] \quad \frac{df}{d\eta} \frac{d\eta}{dt} = \frac{df}{d\eta} \frac{1}{a} = C[f]$$

$$\sigma_T = 1.1 \times 10^{-5} \text{ } ^\circ K$$

$$\frac{\partial f}{\partial \eta} d\eta + \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial p} dp + \frac{\partial f}{\partial n^i} dn^i$$

$$\frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial x^i} \frac{dx^i}{d\eta} + \frac{\partial f}{\partial p} \frac{dp}{d\eta} + \frac{\partial f}{\partial n^i} \frac{dn^i}{d\eta}$$

$$\eta^i (1 - \Phi - \Psi) \quad \frac{1}{p} \frac{dp}{d\eta} = -H \left(-\frac{\partial \Phi}{\partial \eta} - \eta^i \frac{\partial \Psi}{\partial x^i} \right) \quad \text{geodesic eq. for } \gamma$$

$$ds^2 = a^2(\eta) [-(1+2\Psi) d\eta^2 + (1+2\Phi) \delta_{ij} dx^i dx^j]$$

$$g_{00} = -(1+2\Psi) \quad g_{ij} = 1+2\Phi$$

$$\mu\nu - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^4} T^{\mu\nu}$$

$$\frac{df}{d\eta} = -P \frac{\partial f_0}{\partial p} \left(\frac{\partial \theta}{\partial \eta} + \eta^i \frac{\partial \theta}{\partial x^i} + \frac{\partial \Phi}{\partial \eta} + \eta^i \frac{\partial \Psi}{\partial x^i} \right)$$

$$C[f] = -P \frac{\partial f_0}{\partial p} n_e a_T (\theta_0 - \theta + v_b \cdot \vec{n})$$

$$\frac{\partial \theta}{\partial \eta} + \eta^i \frac{\partial \theta}{\partial x^i} + \frac{\partial \Phi}{\partial \eta} + \eta^i \frac{\partial \Psi}{\partial x^i} = \underbrace{n_e a_T a}_{-\dot{c}} (\theta_0 - \theta + v_b \cdot \vec{n})$$

$$\frac{\partial}{\partial x} \rightarrow i k \quad \mu \equiv \frac{\vec{k} \cdot \vec{n}}{k} \quad \eta^i \frac{\partial}{\partial x^i} \rightarrow \eta^i \cdot (i k) = i \mu k$$

$$\dot{\theta} + i \mu k \theta + \dot{\Phi} + i k \mu \Psi = -\dot{c} (\theta_0 - \theta + \mu v_b)$$

1st-order Boltzmann eq. for γ in Fourier space

$$ik\mu - \dot{z})\theta = -\dot{\Phi} - ik\mu\psi - \dot{z}(\theta_0 + \mu v_b)$$

$$-\dot{\Phi} - ik\mu\psi - \dot{z}(\theta_0 + \mu v_b)$$

$$(ik\mu\eta - z) = \dot{\theta}e' + \theta ik\mu e' + \theta e'(-\dot{z})$$

$$e^{ik\mu\eta - z} = \dot{\theta} + ik\mu\theta - \dot{z}\theta = \dot{\theta} + (ik\mu - \dot{z})\theta = S$$

$$e^{ik\mu\eta - z} = S e^{ik\mu\eta - z}$$

$$\theta e^{ik\mu\eta - z} \Big|_0^{\eta_0} = \int_0^{\eta_0} d\eta S e^{ik\mu\eta - z}$$

$$\theta(\eta_0) e^{ik\mu\eta_0 - \frac{z(\eta_0)}{\tau}} - \underbrace{\theta(0)}_{\rightarrow 0} e^{ik\mu\eta_0 - \frac{z(0)}{\tau}} = \int_0^{\eta_0} d\eta S e^{ik\mu\eta - z}$$

$$\theta(\eta_0) e^{ik\mu\eta_0} = \int_0^{\eta_0} d\eta S e^{ik\mu\eta - z}$$

$$\theta(\eta_0) = \int_0^{\eta_0} d\eta S e^{ik\mu(\eta - \eta_0) - z}$$

$$\theta(\eta_0) = \int_0^{\eta_0} d\eta S e^{ik\mu(\eta - \eta_0) - z}$$

$$= \int_0^{\eta_0} d\eta [-\dot{\Phi} - ik\mu\psi - \dot{z}(\theta_0 + \mu v_b)] e^{ik\mu(\eta - \eta_0) - z}$$

$$\mu = \frac{-1}{ik} \frac{\partial}{\partial \eta} \quad e^z \quad g \equiv -\dot{z} e^{-z}$$

$$\theta(\eta_0) = \int_0^{\eta_0} d\eta [-e^z \dot{\Phi} - \dot{z} e^z \theta_0 - \underbrace{\mu}_{\frac{-1}{ik} \frac{\partial}{\partial \eta}} (ik\psi e^z + v_b e^z)] e^{ik\mu(\eta - \eta_0)}$$

$$= \int_0^{\eta_0} d\eta [-e^z \dot{\Phi} - \dot{z} e^z \theta_0 + \frac{1}{ik} \frac{\partial}{\partial \eta} (ik\psi e^z) + \frac{1}{ik} \frac{\partial}{\partial \eta} (v_b e^z)] e^z$$

$$= \int_0^{\eta_0} d\eta [-e^z \dot{\Phi} - \dot{z} e^z \theta_0 + \underbrace{\psi e^z}_{\rightarrow g} + \psi e^z (-\dot{z}) + \frac{1}{k} \frac{\partial}{\partial \eta} (-iv_b e^z)] e^z$$

$$= \int_0^{\eta_0} d\eta [(\theta_0 + \psi)g + \frac{\partial}{\partial \eta} (\frac{iv_b g}{k}) + e^z (\dot{\psi} - \dot{\Phi})] e^{ik\mu(\eta - \eta_0)}$$

$$\int_{-1}^1 \frac{d\mu}{2} P_l(\mu) \theta(\eta_0) = \int_0^{\eta_0} d\eta \bar{S} \int_{-1}^1 \frac{d\mu}{2} P_l(\mu) e^{ik\mu(\eta - \eta_0)} = \int_0^{\eta_0} d\eta \bar{S} j_l$$

$$(-i)^l \theta_l(\eta_0) = \frac{1}{(-i)^l} \int_0^{\eta_0} d\eta \bar{S} j_l(k(\eta - \eta_0))$$

$$\Theta_l(\eta_0) = (-1)^l \int_0^{\eta_0} d\eta \bar{\zeta} j_l(k(\eta - \eta_0))$$

$$\Theta_l(\eta_0) = \int_0^{\eta_0} d\eta \bar{\zeta} j_l(k(\eta_0 - \eta))$$

$$\Theta_l(\eta_0) = \int_0^{\eta_0} d\eta \left[(\theta_0 + \psi) g + \frac{\partial}{\partial \eta} \left(\frac{i g v_b}{k} \right) + \bar{e}^z (\dot{\psi} - \dot{\Phi}) \right] j_l(k(\eta_0 - \eta))$$

$$\int_0^{\eta} d\eta \frac{\partial}{\partial \eta} \left(\frac{i g v_b}{k} \right) j_l(k(\eta_0 - \eta)) = \left[\frac{i g v_b}{k} j_l(k(\eta_0 - \eta)) \right]_0^{\eta_0} - \int_0^{\eta} d\eta \left(\frac{i v_b g}{k} \right) \frac{\partial}{\partial \eta} j_l(k(\eta_0 - \eta))$$

$(ab)' = a'b + ab' \rightarrow \int ab' = [ab] - \int a'b$
 $g(\eta_0) = g(0) = 0$
 $\hookrightarrow \underbrace{j_{l-1}(k(\eta_0 - \eta)) - \frac{l+1}{k(\eta_0 - \eta)} j_l(k(\eta_0 - \eta))}_{\text{...}}$

$$\Theta_l(\eta_0) = [\theta_0(\eta_*) + \psi(\eta_*)] j_l(k(\eta_0 - \eta_*)) - i v_b \left[j_{l-1}(k(\eta_0 - \eta_*)) - \frac{l+1}{k(\eta_0 - \eta_*)} j_l(k(\eta_0 - \eta_*)) \right]$$

$+ \int_0^{\eta_0} d\eta \bar{e}^z (\dot{\psi} - \dot{\Phi}) j_l(k(\eta_0 - \eta))$
 $v_b = 3\dot{\lambda}\theta_0$

$$\Theta_l(\eta_0) = [\theta_0(\eta_*) + \psi(\eta_*)] j_l(k(\eta_0 - \eta_*)) + 3\theta_0 \left[j_{l-1}(k(\eta_0 - \eta_*)) - \frac{l+1}{k(\eta_0 - \eta_*)} j_l(k(\eta_0 - \eta_*)) \right] + \int_0^{\eta_0} d\eta \bar{e}^z (\dot{\psi} - \dot{\Phi}) \times j_l(k(\eta_0 - \eta))$$

$$\mathcal{L} \rightarrow \ddot{\phi} \rightarrow \phi + \delta\phi \rightarrow \delta\ddot{\phi} \rightarrow \ddot{h} \rightarrow \delta\phi \rightarrow \xi \rightarrow \psi \rightarrow P_k \rightarrow \delta \rightarrow P(k)$$

$$C_\ell(\eta) = \frac{2}{\pi} \int_0^{\eta_0} dk k^2 P(k) \left| \frac{\theta_\ell(k)}{\delta(k)} \right|^2$$

$$2 \frac{\dot{a}}{a^3} \dot{\phi} + \frac{1}{a^2} \ddot{\phi} - \frac{1}{a^2} \nabla^2 \phi + V'(\phi) = 0$$

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} - V(\phi) \quad \phi \text{ inflation field}$$

$V(\phi)$ potential of ϕ

$$\ddot{\phi} + 2aH\dot{\phi} - \nabla^2 \phi + a^2 V'(\phi) = 0$$

$$\phi \rightarrow \phi + \delta\phi \quad V(\phi + \delta\phi) = V(\phi) + V'(\phi)\delta\phi$$

$$\delta\ddot{\phi} + 2aH\delta\dot{\phi} - \nabla^2 \delta\phi + a^2 V''(\phi)\delta\phi = 0$$

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

Euler-Lagangian eq. $\frac{\partial \mathcal{L}}{\partial \phi} = -V'(\phi)$

$$h \equiv a\delta\phi$$

$$\nabla \rightarrow ik$$

$$g_{\mu\nu} = a^2(\eta) \text{diag}[-1, 1, 1, 1] \quad \text{FRW metric}$$

$$g_{\mu\nu} = \begin{bmatrix} -a^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{bmatrix}$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = -g^{\mu\nu} \partial_\nu \phi$$

$$\begin{aligned} \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) &= -\partial_\mu g^{\mu\nu} \partial_\nu \phi - g^{\mu\nu} \partial_\mu \partial_\nu \phi \\ &= -\partial_0 g^{00} \partial_0 \phi - g^{00} \partial_0 \partial_0 \phi - \frac{1}{a^2} \nabla^2 \phi \\ &= 2 \frac{\dot{a}}{a^3} \dot{\phi} + \frac{1}{a^2} \ddot{\phi} - \frac{1}{a^2} \nabla^2 \phi \end{aligned}$$

$$\ddot{h} + \left(k^2 + \frac{a^2 V''}{\eta^2} - \frac{\ddot{a}}{a} \right) h = 0 \quad \epsilon \equiv \frac{d}{dt} \left(\frac{1}{H} \right)$$

$$\frac{3\delta}{\eta^2} \quad \frac{2+3\epsilon}{\eta^2} \quad \sigma \equiv \frac{1}{8\pi G} \frac{V''}{V}$$

$$\ddot{h} + \left[k^2 + \frac{1}{\eta^2} (3\delta - 2 - 3\epsilon) \right] h = 0$$

$$\ddot{h} + \left[k^2 + \frac{1}{\eta^2} \left(\frac{1}{4} - \nu^2 \right) \right] h = 0 \quad \nu^2 = \frac{9}{4} - 3\delta + 3\epsilon$$

$$h = \sqrt{-\eta} \left[H_\nu^{(1)}(-k\eta) + H_\nu^{(2)}(-k\eta) \right]$$

$$h \simeq \frac{e}{\sqrt{2k}}$$

$$\delta\phi = \frac{e}{\sqrt{2k}}$$

$$|\delta\phi| = \sqrt{2}$$

$$\xi \equiv$$

$$\approx \frac{e^{\lambda(\nu-\frac{1}{2})\pi}}{\sqrt{2k^3}} aH \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu}$$

$$\psi = -\frac{2}{3} \xi = -\frac{2}{3} \left(\frac{aH}{\dot{\phi}}\right) \delta\phi$$

$$\delta = \frac{3k^2}{5H_0^2 \Omega_{m0}} T(k) D(a) \Phi_p$$

$$\hookrightarrow \langle \Phi_p \Phi_p^* \rangle = P_\delta = P_\psi$$

$$\delta\phi = \frac{e^{\lambda(\nu-\frac{1}{2})\pi}}{\sqrt{2k^3}} H \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu}$$

$$P_\psi = \frac{4}{9} \left(\frac{aH}{\dot{\phi}}\right)^2 |\delta\phi|^2$$

$$= \frac{4}{9} \frac{4\pi G}{\epsilon} \frac{H^2}{2k^3} \left(\frac{k}{aH}\right)^{3-2\nu}$$

$$P_\psi = P_\delta = \frac{50\pi^2}{9k^3} \left(\frac{k}{aH}\right)^{n-1} \sigma_H^2 \frac{\Omega_{m0}^2}{D^2(a_0)}$$

$$|\delta\phi| = \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu}$$

$$= \frac{8\pi G}{9k^3} \frac{H^2}{\epsilon} \left(\frac{k}{aH}\right)^{n-1}$$

$3-2\nu = 2\delta-2\epsilon$
 $= n-1$
 Harrison-goldovich-fukukawa spectrum

$$P(k) = \langle \delta \cdot \delta^* \rangle = \frac{9k^4}{25H_0^4 \Omega_{m0}^2} T^2(k) D^2(a) \langle \Phi_p \rangle$$

$$\xi \equiv -\frac{i k_i \delta T_i^0 H}{k^2 (p+\rho)} - \psi$$

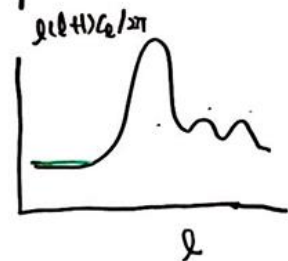
$$\xi = \frac{aH}{\dot{\phi}} \delta\phi$$

$$p+\rho = \frac{4}{3}\rho_r \quad \theta_i = \frac{k\psi}{6aH} \quad \left(\frac{aH}{\dot{\phi}}\right)^2 = \frac{4\pi G}{\epsilon}$$

$$i k_i \delta T_i^0 = 4a k \rho_r \theta_i$$

$$\xi = -\frac{4a k \rho_r \left(\frac{k\psi}{6aH}\right) H}{k^2 \frac{4}{3}\rho_r} - \psi$$

$$= -\frac{1}{2}\psi - \psi = -\frac{3}{2}\psi$$



$$= \frac{9k^4}{25H_0^4 \Omega_{m0}^2} T^2(k) D^2(a) \frac{50\pi^2}{9k^3} \left(\frac{k}{aH}\right)^{n-1} \sigma_H^2 \frac{\Omega_{m0}^2}{D^2(a_0)}$$

$$= 2\pi^2 \sigma_H^2 \frac{k^n}{H_0^{n+3}} \left(\frac{D(a)}{D(a_0)}\right)^2 T^2(k)$$

$$C_\ell = \langle \dots \rangle$$

$$l(l+1) C_\ell^{LS}(\eta) = \frac{\pi \Omega_{m0}^2}{2 D^2(a_0)} \sigma_H^2$$

$$C_\ell = \int \frac{d^3k}{(2\pi)^3}$$

$$\times \int d\eta$$

$$C_\ell = \frac{2}{\pi} \int dk k^2 P(k) \left| \frac{\theta_\ell(k)}{\delta(k)} \right|^2$$

$$C_\ell = \int$$

$$\theta(\vec{x}, \vec{n}, \eta) \equiv \frac{d\Gamma}{T}$$

$$\theta(\vec{k}, \vec{n}) = \sum_{l=0}^{\infty} (-i)^l (2l+1) \underline{P_l(\vec{k} \cdot \vec{n})} \underline{\theta_l(k)}$$

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4

$$\theta(\vec{x}, \vec{n}, \eta) = \sum_{l=1}^{\infty} \sum_{m=-l}^l a_{lm} Y_{lm}(\vec{n})$$

$$\int d\Omega \theta \cdot Y_{lm}^*(\vec{n}) = \sum_{l'} \sum_{m'} a_{l'm'} \underbrace{\int d\Omega Y_{l'm'}(\vec{n}) Y_{lm}^*(\vec{n})}_{\delta_{ll'} \delta_{mm'}} = a_{lm}$$

$$\Gamma^2(k) D^2(a) \langle \Phi_p \Phi_p^* \rangle \quad a_{lm} = \int d\Omega \theta(\vec{x}, \vec{n}, \eta) Y_{lm}^*(\vec{n})$$

$$\left(\frac{k}{\lambda H_0}\right)^{n+1} \delta_H^2 \frac{\Omega_{m0}}{D^2(a_p)} \quad a_{lm} = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} \int d\Omega \theta Y_{lm}^* \quad a_{lm}^* = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k} \cdot \vec{x}} \int d\Omega \theta^* Y_{lm}$$

$$C_l = \langle a_{lm} a_{lm}^* \rangle = \int \frac{d^3k}{(2\pi)^3} e^{+i\vec{k} \cdot \vec{x}} \int \frac{d^3k'}{(2\pi)^3} e^{-i\vec{k}' \cdot \vec{x}} \langle \theta(k, n) \theta^*(k', n) \rangle \int d\Omega Y_{lm}^* \int d\Omega Y_{lm}$$

$$C_l = \int \frac{d^3k}{(2\pi)^3} \int d^3k' \frac{P(k)}{|f(k)|^2} \sum_{l' l''} \sum_{m' m''} (-i)^{l'} (i)^{l''} \underbrace{(2l'+1)(2l''+1) \delta_{ll'} \delta_{mm'}}_{\theta_l \theta_l^*} \underbrace{\langle \delta \cdot \delta^* \rangle}_{(2\pi)^2 \delta(k-k') P(k)} \frac{\theta(k) \theta^*(k')}{\delta \cdot \delta^*}$$

$\underbrace{\int d\Omega P_l(\vec{k} \cdot \vec{n}) Y_{lm}}_{\left(\frac{4\pi}{2l+1}\right) Y_{lm}} \underbrace{\int d\Omega P_l(\vec{k} \cdot \vec{n}) Y_{lm}^*}_{\left(\frac{4\pi}{2l+1}\right) Y_{lm}^*}$

$$d^3k = dk k^2 d\Omega$$

$$C_l = \int \frac{d^3k}{(2\pi)^3} \left| \frac{\theta_l(k)}{\delta(k)} \right|^2 (4\pi)^2 P(k) Y_{lm} Y_{lm}^* = \frac{2}{\pi} \int dk k^2 \left| \frac{\theta_l(k)}{\delta(k)} \right|^2 P(k) \underbrace{\int d\Omega Y_{lm} Y_{lm}^*}_{1}$$